

# THE BULLET PROBLEM WITH DISCRETE SPEEDS

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ABSTRACT. Bullets are fired, one per second, along the positive real line with independent and identically distributed speeds. Collisions result in mutual annihilation. For bullet speeds sampled uniformly from any finite set, we establish a phase transition for survival of the first bullet. We also exhibit a family of continuous speed distributions where, if the first bullet has speed in the lower half of the distribution, then it is almost surely annihilated.

## 1. INTRODUCTION

The bullet process is among the simplest of annihilating ballistic systems. Such systems have been around since the 1970's, but very little progress has been made. The combination of simplicity and difficulty make the bullet problem compelling and, according to many, rather addictive. It can be described in three sentences: Each second, a bullet is fired from the origin along the positive real line with a speed uniformly sampled from  $(0, 1)$ . When a faster bullet collides with a slower one, they mutually annihilate. Is the probability the first bullet survives strictly between 0 and 1? Though our results do not include the case of  $\text{Uniform}(0,1)$  speeds, our main theorem suggests the conjectured phase transition in the classical bullet problem and lays out an approach for proving this.

We study two variants that modify the speed distribution of the bullets. In one, the example to keep in mind is when bullet speeds are uniformly selected from a discrete set. For this, and slightly more general speed distributions, we establish a phase transition for survival of the first bullet. In another variant, we provide a class of continuous speed distributions for which the first bullet perishes with probability at least one half.

To our knowledge, this is the first published work on the bullet process. We learned near the end of this project our Theorem 4 (that for certain distributions the slower half of bullets perish) was known to Vladas Sidoravicius and Laurent Tournier. Their result is unpublished and is for an equivalent process they introduced known as the *arrow process*. More details are in Section 1.3.

**1.1. Theorem statements.** Let us introduce some notation and define the bullet process a bit more formally. Consider bullets  $\{b_1, b_2, \dots\}$  fired from the origin along the real line such that  $b_{i+1}$  is fired one second after  $b_i$  for all  $i \geq 1$ . This delay between firings is not so important. All of the results here also hold for exponentially distributed firing times.

The speed of bullet  $b_i$  is denoted by  $s(b_i)$ . The bullets have independent and identically distributed (i.i.d.) speeds. We assume that the speeds are obtained from a probability space  $(S, \mu)$  with  $S \subseteq (0, \infty)$  and  $\mu$  a probability measure on  $S$ . When two or more bullets collide, all of them are annihilated. We will refer to this as an  $(S, \mu)$ -bullet process.

Let  $b_i \mapsto b_j$  denote the event of bullet  $b_i$  colliding into  $b_j$ , thus resulting in their mutual annihilation. We say that  $b_i$  catches  $b_j$ . Note that this can only happen if  $i > j$  and  $s(b_i) > s(b_j)$ . Define  $\tau$  to be the minimum index with  $b_\tau \mapsto b_1$ . The minimum is to account

for the possibility of a simultaneous collision of several bullets. If  $b_1$  is never caught by another bullet, set  $\tau = \infty$ . When  $\tau = \infty$ , we say that  $b_1$  *survives*. When  $\tau < \infty$ , we say that  $b_1$  *perishes*. Our main result is a non-trivial phase transition when the bullet speeds are uniformly sampled from a finite set.

**Theorem 1.** *Suppose bullet speeds are uniformly sampled from  $\{s_n, s_{n-1}, \dots, s_1\} \subseteq (0, \infty)$  with  $s_n < s_{n-1} < \dots < s_2 < s_1$  arbitrary, but fixed.*

(i) *If the first bullet has the second fastest speed, then it survives with positive probability:*

$$\mathbf{P}[b_1 \text{ survives} \mid s(b_1) = s_2] > 0.$$

(ii) *If the first bullet has the slowest speed, then it perishes almost surely:*

$$\mathbf{P}[b_1 \text{ perishes} \mid s(b_1) = s_n] = 1.$$

Note that survival of  $b_1$  when it has maximal speed is trivial. The canonical coupling (that aligns  $s(b_i)$  for  $i \geq 2$ ) guarantees that the probability the first bullet survives is monotonically increasing with respect to its speed. This monotonicity, along with Theorem 1 imply that there is a speed  $s_{i^*}$  at which an initial bullet slower than  $s_{i^*}$  will perish, while one with speed at least  $s_{i^*}$  will survive with positive probability. So,  $n \leq i^* \leq 2$ . Possibly  $i^*$  depends on the choice of the  $s_i$ .

Our main result is a consequence of two theorems about survival and perishing. The hypotheses do not require  $S$  to be totally discrete. For example, we can also deduce a phase transition when  $S = \{1, 2, 3, 4\} \cup (2, 3)$  with the uniform measure on  $(2, 3)$ . We chose to state Theorem 1 in a concrete way for the sake of clarity.

**Theorem 2.** *Suppose that  $S = S' \cup \{s_2, s_1\}$  where  $S' \subseteq (0, s_2)$  and  $0 < s_2 < s_1 < \infty$ . If  $\mu(S') > 0$  and  $\mu(\{s_2\}) \geq \mu(\{s_1\})$ , then*

$$\mathbf{P}[b_1 \text{ survives} \mid s(b_1) = s_2] > 0.$$

Theorem 2 says that there are many measures on processes with two fastest speeds in which the second fastest bullet survives. A counterpart for slow speeds holds: when there are two slowest speeds, then a bullet with the slowest speed that is fired first perishes a.s.

**Theorem 3.** *Suppose that  $S = S'' \cup \{t_2, t_1\}$  where  $S'' \subseteq (t_1, \infty)$  and  $0 < t_2 < t_1$ . If  $\mu(S'') > 0$  and  $\mu(\{t_1\}) \geq \mu(\{t_2\})$ , then*

$$\mathbf{P}[b_1 \text{ survives} \mid s(b_1) = t_2] = 0.$$

The proof of Theorem 3 uses Theorem 2. With this technique, it is important that Theorem 2 applies to arbitrary speed sets,  $S$ , with a second largest element. With a general theorem for survival of, say, a third fastest bullet, our same argument would imply the second slowest bullet perishes.

An extension of the idea in the proof of Theorem 3 shows that, for certain speed distributions, the probability the first bullet perishes is at least  $1/2$ . The uniform distribution is not covered by our theorem, but we have this result for a fairly close approximation. First we state a general theorem, then give the example.

**Theorem 4.** *Suppose that  $S = [s_n, s_1]$  with  $0 < s_n < s_1 < \infty$ , and let  $\mu$  be any probability measure on  $S$  satisfying*

$$(1) \quad \mu([s_n, x]) = \mu([T(x), s_1]), \quad \forall x \in S,$$

with  $T(x) = (s_n^{-1} + s_1^{-1} - x^{-1})^{-1}$ . In such an  $(S, \mu)$ -bullet process,

$$\mathbf{P}[b_1 \text{ survives} \mid s(b_1) < s^*] = 0,$$

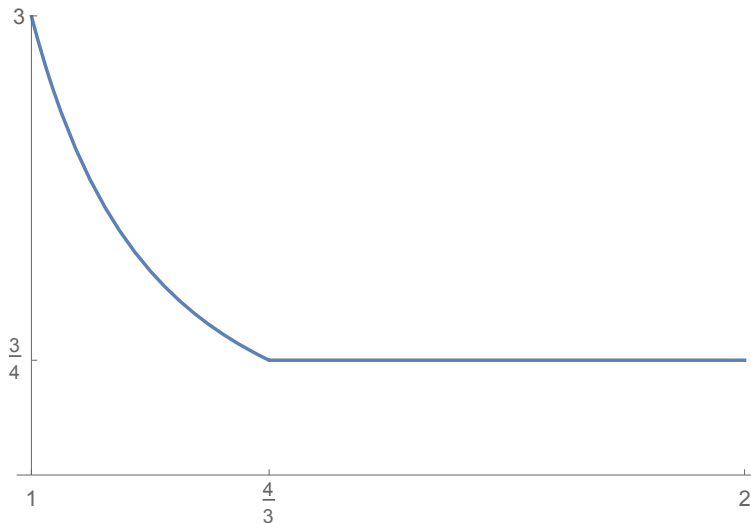


FIGURE 1. The density function  $f$  from Example 5. A bullet process with speeds sampled according to  $f$  has bullets slower than  $4/3$  perishing a.s.

where  $s^* = \frac{2s_1s_n}{s_1+s_n}$  is the unique fixed point of  $T$ .

Above we have  $\mu([s_n, s^*]) = \mu(s^*, s_1)$ , and thus  $\mu$  places half of its mass on speeds less than or equal to  $s^*$ . The statement of Theorem 4 is not so enlightening on its own. It is best explained via an example.

**Example 5.** Let  $S = [1, 2]$  so that  $s^* = 4/3$ . We will specify  $\mu$  by a density function  $f$  so that  $\mu(A) = \int_A f(x)dx$ . Say we want  $f$  to be uniform on  $[4/3, 2]$ . The requirement that  $\mu[s^*, 1] = 1/2$  gives  $f(x) = 3/4$  on this interval. We can now use (1) along with the fundamental theorem of calculus to deduce the remaining values of  $f$ :

$$f(x) = -f(T(x))T'(x) = \frac{3}{(2-3x)^2}, \quad x \in [1, 4/3).$$

By Theorem 4, the first bullet in a  $([1, 2], \mu)$ -bullet process perishes a.s. if  $s(b_1) < 4/3$  (see Figure 1).

Besides resembling the classical bullet problem, there is nothing particularly special about  $[1, 2]$  or the appearance of the uniform measure on  $[4/3, 2]$  in the above example. One could choose any distribution on  $[s_n, s^*]$  or  $[s^*, s_1]$ , and (1) would dictate the other half of the distribution.

**1.2. Overview of proofs.** The idea for the proof of Theorem 2 is that  $\tau$  can be recursively related back to independent copies of itself. In Proposition 8, we show that if  $s(b_1) = s_2$  and  $s(b_2) = s_2$  as well, then  $b_1$  survives “twice” as long as it would have otherwise. Also, if  $s(b_1) = s_2$  and the second bullet is slower than  $s_2$ , then  $b_2$  acts as a shield for  $b_1$ —thus increasing the survival time of  $b_1$ . These arguments hinge on the renewal properties described in Lemma 6 and Lemma 7.

We then obtain a recursive distributional equation. The analysis from here is inspired by the approach used in [HJJ15b, HJJ15a, JJ16] to prove that the frog model on trees is recurrent. It also relates to an elementary argument that the return time of a biased random

walk is infinite with positive probability. This is described at the beginning of Section 2. The idea is to transform into a recursion with probability generating functions. In Proposition 8, we obtain the generating function for some  $\tau^* \preceq \tau$  and use it to show that  $\tau^*$ , and thus also  $\tau$ , is infinite. We do this in Section 2.

To prove Theorem 3 and Theorem 4, we transform the bullet problem to an arrow process where all particles start at the same time (see Lemma 12 and Lemma 13). After extending to all of  $\mathbb{Z}$ , the arrow process is ergodic. When we assume a slow bullet survives with positive probability, we can use the Birkhoff ergodic theorem along with monotonicity to obtain a contradiction. This is done in Section 3.

**1.3. Historical discussion and further questions.** The bullet process is an annihilating ballistic system. The study of such systems was initiated by Erdős and Ney in [EN74]. They considered the transience-recurrence behavior of annihilating simple random walks started from each integer. Arratia reinitiated the study in one dimension, and also proved results in higher dimension (see [Arr81, Arr83]). More recently, annihilating Brownian motion has been studied ([TZ11]). The bullet process is less random than these processes. Curiously, the amount of randomness, and the difficulty to prove asymptotic results appear to be inversely related.

The inverse relationship is well-illustrated by *annihilating non-backtracking random walks*. This process modifies the law for particles' paths in annihilating simple random walk to be non-backtracking. That is, each particle is barred from crossing an edge already included in its range. Notice that particles are more tethered to their past. Ballistic systems, like the bullet process and those described below, are more extreme versions of this tethering. This is the main source of difficulty in analyzing these processes.

Currently we can say very little about the forefather (annihilating non-backtracking random walk) nor its offspring (annihilating ballistic systems). While annihilating simple random walks are known to be recurrent on  $\mathbb{Z}^d$  (see [Arr83]), this question is open for the non-backtracking variant. In fact, understanding the long time behavior is open for any transient graph. We learned of this question from Itai Benjamini, but its origins are unclear.

**Open Question 1.** *Exhibit a transient graph on which non-backtracking annihilating random walk is recurrent.*

At the time of writing, there are no published papers on the bullet problem. This dearth of results is not due to lack of interest—as many researchers have spent time on it—but rather to the difficulty to prove anything. The problem has been shared widely, but mostly via word of mouth. This is our best attempt to record the history and current state of the problem.

The IBM problem of the month of May in 2014 credits a version of the problem to David Wilson. The question there is to fire exactly  $2m$  bullets with independent uniform(0, 1) speeds and compute the probability that not a single bullet survives. There is an unpublished result of Fedja Nazarov that this probability is  $\prod_{i=1}^m 1 - \frac{1}{2^i}$ . [Private correspondence with Yuval Peres].

The infinite case is trickier: very few results are known. Kostya Makarychev has, in aggregate, simulated over a 100 billion bullets. These computations led to the conjecture that, when the speeds are uniform(0, 1), the first bullet survives with positive probability if its speed is larger than  $\approx .9$ . Theorem 2 is a first step towards proving such a result. The next step will be to prove that slower bullets can also survive with positive probability.

**Open Question 2.** *Suppose that bullets have speeds  $0 < s_n < s_{n-1} < \dots < s_1 < \infty$  sampled uniformly. Show, for large enough  $n$ , that if the initial bullet has speed  $s_{\lfloor .01n \rfloor}$ , then it survives with positive probability.*

A theorem along the lines of Open Question 2 would also imply that bullets slower than  $s_{\lfloor .99n \rfloor}$  perish. This would follow along the lines of the proof of Theorem 3. From there, it may be possible to interpolate to the uniform distribution. For example, bullet speeds could be the dyadic numbers in  $[0, 1]$ . Even proving that  $b_1$  survives when  $s(b_1) = s_3$  in the above question would be worthwhile.

As previously mentioned, some (unpublished) progress has been made on a related model known as the *arrow process* introduced by Laurent Tournier and Vladas Sidoravicius. It starts by placing arrows on  $\mathbb{Z}$ . They are then shot simultaneously at different speeds, traveling along the real line until a collision occurs. This results in mutual annihilation.

Fix some  $p < 1$  and assign to each arrow an i.i.d. speed. The canonical example here is speeds in  $\{-1, 0, 1\}$  with 0 being assigned with probability  $p$ , and speeds  $-1$  and  $1$  each with probability  $(1 - p)/2$ . Tournier and Sidoravicius can show for sufficiently large  $p$  that an arrow with speed 0 survives with positive probability. Lorenzo Taggi also has an unpublished proof of this for  $p > .31$ . The problem generating the most interest in this process is to show that, for a small enough  $p$ , an arrow with speed 0 also perishes a.s. Simulations suggest that this happens when  $p < 1/4$  [private communication with Laurent Tournier, Vladas Sidoravicius, and Lorenzo Taggi].

**Open Question 3.** *In the  $\{-1, 0, 1\}$  arrow process, show that if the probability of a speed-0 arrow is sufficiently small, then every arrow perishes a.s.*

Ergodicity and symmetry in the  $\{-1, 0, 1\}$  arrow process ensure that every speed  $-1$  and  $1$  arrow perishes a.s. We describe the proof of this already well-known fact in more generality in Proposition 14. The equivalence between the annihilations in the bullet process and those in a one-sided arrow process is described in Lemma 12 and Lemma 13. We piece these three statements together to prove Theorem 4.

There are also higher-dimensional versions of the bullet problem. One formulation in  $\mathbb{R}^d$  places an  $\epsilon > 0$  ball around each bullet (or cannon ball, rather). A ball is fired from the origin in  $\mathbb{R}^d$  every second in a uniformly random direction with, say,  $\text{uniform}(0, 1)$  speeds.

**Open Question 4.** *Are there values  $d \geq 2$  and  $\epsilon > 0$  such that the first cannon ball survives with positive probability?*

Another related problem is the meteor problem which is attributed to Itai Benjamini. Place  $\epsilon$ -balls (meteors) in Euclidean space with centers according to a unit intensity Poisson process. At time 0, each meteor shoots off in a direction chosen uniformly at random. All meteors have the same speed. When two meteors collide (when their centers come within  $2\epsilon$  of each other), they annihilate. Further discussion and partial progress for a related process on trees is in [BFGG<sup>+</sup>16]. Say that the origin is occupied whenever the center of a meteor is within  $\epsilon$  of it.

**Open Question 5.** *Is the origin a.s. occupied by a meteor?*

The elusive nature of these simple processes—bullets, arrows, cannon balls, meteors—suggests shortcomings in our toolbox for working on annihilating systems. The approach used here advances our understanding of the bullet problem, and, hopefully, will extend to these related processes.

**1.4. Acknowledgements.** We thank Omer Angel for initially sharing the classic bullet problem with us at the PIMS Stochastics Workshop at BIRS in September 2015. Toby Johnson provided a nice reference that connected part of the proof to an argument with random walks. Itai Benjamini, Alexander Holroyd, Yuval Peres, Vladas Sidoravicius, Alexandre Stauffer, Lorenzo Taggi, and David Wilson were helpful in understanding the folklore surrounding the problem. Many thanks to Laurent Tournier for a careful reading and helpful feedback. Rick Durrett, Jonathan Mattingly, Jim Nolen, and the students in Fall semester 2016 of Math 690-40 at Duke University gave very useful feedback when these results were presented. The undergraduates on this paper were partially supported by the 2016 University of Washington Research Experience for Undergraduates.

## 2. SURVIVAL OF FAST BULLETS

Throughout this section, we assume that speeds come from a probability space  $S = S' \cup \{s_1, s_2\}$  as in Theorem 2. The first step of the proof relates  $\tau$ , the minimum index of the bullet that catches  $b_1$  conditional on  $s(b_1) = s_2$ , to independent copies of itself. We then use probability generating functions to show that any random variable satisfying the relationship between  $\tau$  and its copies must be infinite with positive probability.

As a warmup to our proof, we use a similar technique to show that the return time,  $\kappa$ , to the origin of a biased random walk starting at 1 on the integers is infinite with positive probability. Let  $s(b_1)$  be the position of the walk after one step, and suppose  $s(b_1) = 2$  with probability  $p > 1/2$ . Conditioning on the first step, we can write

$$\kappa \stackrel{d}{=} \mathbf{1}\{s(b_1) = 0\} + \mathbf{1}\{s(b_1) = 2\}(\kappa_1 + \kappa_2 + 1),$$

where  $\kappa_1$  and  $\kappa_2$  are i.i.d. copies of  $\kappa$  and  $\mathbf{1}\{\cdot\}$  is an indicator function. When  $s(b_1) = 2$ , the walk must return to 1, then return to 0. Each takes an independent  $\kappa$ -distributed amount of steps.

To show that  $\kappa$  is infinite with positive probability, we introduce the probability generating function  $g(x) = \mathbf{E}x^\kappa$ . The distributional relationship with  $\kappa, \kappa_1$ , and  $\kappa_2$  implies that  $g(x) = (1-p)x + pxg(x)^2$ . Solving for  $g(x)$ , and taking the negative root to ensure  $g(0) = 0$ , we have the closed form

$$g(x) = \frac{1 - \sqrt{1 - 4p(1-p)x^2}}{2px}.$$

The series representation,  $g(x) = \sum \mathbf{P}[\kappa = j]x^j$ , implies that  $g(1) = \mathbf{P}[\kappa < \infty]$ . It is simple to check with the above formula that  $g(1) = \frac{1-p}{p} < 1$  since  $p > \frac{1}{2}$ .

The approach to Theorem 2 follows this blueprint. However, we have a distributional inequality rather than equality. This requires a more subtle renewal property, and then additional work to push through the argument with generating functions.

**2.1. Obtaining the recursive equation.** We start with two renewal lemmas. The first states that the bullet speeds following a maximal speed bullet are independent of any event involving this bullet.

**Lemma 6.** *If  $b_\gamma \mapsto b_j$  and  $s(b_\gamma) = s_1$  with  $j$  any fixed index, then the random variables  $\{\gamma, s(b_{\gamma+1}), s(b_{\gamma+2}), \dots\}$  are independent.*

*Proof.* The bullet  $b_\gamma$  has the fastest speed, so the bullets behind it do not interfere. Thus the event  $\{b_\gamma \mapsto b_1\}$  depends only on the bullet speeds  $s(b_1), s(b_2), \dots, s(b_\gamma)$ .  $\square$

A longer range renewal property holds for other annihilations where, outside of a particular window, the bullet speeds become independent.

**Lemma 7.** *Let  $E = E(S, s(b_i), s(b_j), i, j) = \{b_i \mapsto b_j, s(b_i), s(b_j)\}$  be the event that  $b_i$  catches  $b_j$  with  $s(b_i)$  and  $s(b_j)$  known. There exists a finite integer  $a = a(s(b_i), s(b_j), i, j)$  such that the bullet speeds  $s(b_{i+a}), s(b_{i+a+1}), \dots$  are independent of the event  $E$ .*

*Proof.* Given  $i, j, s(b_i)$ , and  $s(b_j)$ , let  $a$  be such that a maximal speed bullet fired at time  $i + a$  cannot reach  $b_i$  before  $b_i \mapsto b_j$ . The event  $b_i \mapsto b_j$  is thus unaffected by the bullet speeds  $s(b_{i+a}), s(b_{i+a+1}), \dots$ . The independence claim follows.  $\square$

We can now show that  $\tau$  stochastically dominates an equation involving independent copies of itself. Recall that  $X \succeq Y$  means there is a coupling with marginals  $X' \sim X$  and  $Y' \sim Y$  such that  $X' \geq Y'$  a.s. (see [SS07]).

**Proposition 8.** *Let  $\tau$  be the index of the bullet that catches  $b_1$  conditional on  $s(b_1) = s_2$ . Let  $\tau_1, \dots, \tau_5$  be i.i.d. copies of  $\tau$ . Let  $S_\epsilon \sim \text{Ber}(\epsilon)$  also be independent.*

$$(2) \quad \tau \succeq \mathbf{1}\{s(b_2) < s_2\}(S_\epsilon(\tau_1 + \tau_2) + (1 - S_\epsilon)\tau_3)$$

$$(3) \quad \tau \succeq \mathbf{1}\{s(b_2) = s_2\}(\tau_4 + \tau_5)$$

$$(4) \quad \tau \succeq \mathbf{1}\{s(b_2) = s_1\}$$

*Proof.* We will establish each line of the above in reverse order by conditioning on the value of  $s(b_2)$ . When  $s(b_2) = s_1$  as in (4), we have  $b_2 \mapsto b_1$  deterministically. Although  $\tau = 2$  on this event, it will simplify our calculations later to use the indicator function as a lower bound.

When  $s(b_2) = s_2$  as in (3), suppose that  $b_\sigma$  destroys  $b_2$ . We have translated the original setup by one index, so  $\sigma \sim \tau + 1$ . Only a bullet with fastest speed can destroy  $b_2$ , thus  $s(b_\sigma) = s_1$ . Lemma 6 ensures that the speeds  $s(b_{\sigma+1}), s(b_{\sigma+2}), \dots$  are independent of  $\sigma$ . Suppose that  $b_{\sigma'} \mapsto b_1$ . Once again this is the first unobstructed speed- $s_1$  bullet after  $b_\sigma$ . Thus  $\sigma' - \sigma \sim \tau$ , and this difference is independent of  $\sigma$ . This is where the term  $\tau_4 + \tau_5$  in (3) comes from (see Figure 2).

The pivotal case is (2), when  $s(b_2) < s_2$ . The idea is that  $b_2$  acts as a shield, and causes an  $\epsilon$ -bias for the bullets close behind it to have speed  $s_2$ . To see this rigorously, suppose that  $b_\gamma$  is the first bullet destroying  $b_2$ . First note that if  $\gamma$  is infinite with positive probability, then so is  $\tau$  and our theorem is proven. Accordingly, let us suppose that  $\gamma$  is a.s. finite. Also note that in order for  $b_\gamma \mapsto b_2$  to occur, all of the bullets  $b_3, \dots, b_{\gamma-1}$  must mutually annihilate. We can then ignore them for the remainder of the argument.

When  $s(b_\gamma) = s_1$ , it resets the model just as in the  $s(b_2) = s_2$  case, and  $b_1$  survives until a bullet with index distributed as  $\tau + \gamma$  destroys it. There is also the possibility that  $s(b_2) < s(b_\gamma) < s_1$ . When this occurs, let  $a = a(s(b_\gamma), s(b_2), \gamma, 2)$  be the largest index for which  $b_{\gamma+a}$  could intercept  $b_\gamma$ . More precisely,  $a$  is the largest index such that, if  $s(b_{\gamma+a}) = s_1$ , then the time at which  $b_{\gamma+a}$  could potentially catch  $b_\gamma$  (if uninterrupted) would be earlier than the time of collision of  $b_\gamma$  and  $b_2$ . Bullets with indices in the set  $I = \{\gamma + 1, \dots, \gamma + a\}$  are dependent upon  $s(b_\gamma), s(b_2)$ , and  $\gamma$ . In particular bullets faster than  $s(b_\gamma)$  can survive to intercept  $b_\gamma$ . By Lemma 7, the bullets with indices larger than  $\gamma + a$  are once again independent (see Figure 3).

The model resets after  $\gamma + a$ . Let us restrict our attention to just the bullets with indices in  $I$ . That is, consider a bullet model with only  $|I|$  bullets, with speeds conditioned so that  $b_\gamma \mapsto b_2$  with  $s(b_2) < s_2$ . Since  $b_\gamma \mapsto b_2$ , no bullets with speed  $s_1$  in  $I$  can survive, since such a bullet would catch  $b_\gamma$  before  $b_\gamma$  catches  $b_2$ . As  $a$  is finite, there is a positive probability,  $\epsilon > 0$ , that all of the surviving bullets in  $I$  will have speed  $s_2$ . For example, there is a positive probability of a sequence of alternating between bullets with speeds from  $S'$  (as

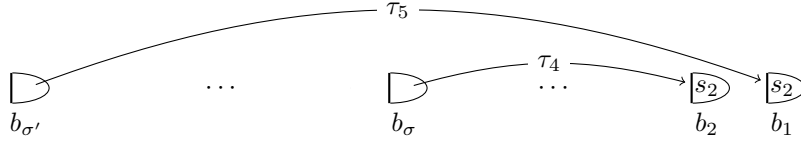


FIGURE 2. The picture when  $s(b_2) = s_2$ . The bullets  $b_2$  and  $b_{\sigma}$  are annihilated by a bullet that is fired a  $\tau$ -distributed number of seconds after it.

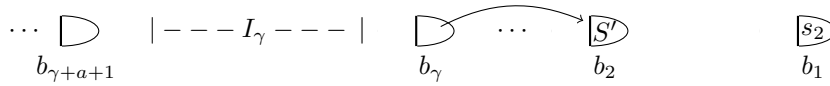


FIGURE 3. The picture when  $s(b_2) \in S'$ . If  $b_{\gamma} \mapsto b_2$ , then there is an interval of bullets behind it that contains no surviving  $s_1$ -speed bullets. With probability at least  $\epsilon$  it contains a surviving speed- $s_2$  bullet. Bullets  $b_{\gamma+a+1}$  onward are i.i.d.

defined in the statement of Theorem 2) and bullets with speed  $s_2$ . When this occurs, we use the argument from (3) to deduce that the index of the destroying bullet of  $b_1$  is at least the sum of two independent copies of  $\tau$ . This corresponds to  $\tau_1 + \tau_2$  in (2).

In all other realizations of the bullet speeds in  $I$ , these slower bullets only prolong the survival of  $b_1$ . This is because we can repeat this argument for the largest index surviving bullet in  $I$ , and again obtain a new window of speed- $s_2$  bullets, or one of slower bullets. Thus, it is monotonically worse to remove all of the bullets in  $I$ , and restart the model with  $s(b_1) = s_2$  and setting  $s(b_i) = s(b_{\gamma+a+i})$  for  $i \geq 2$ . The survival of  $b_1$  in this setting is again distributed as  $\tau$ . This corresponds to  $\tau_3$  in (2). We then attach the Bernoulli random variable  $S_{\epsilon}$  to the event that all of the surviving bullets in  $I$  have speed  $s_2$ . Otherwise, we have established that  $b_1$  survives at least a  $\tau$  distributed amount of time.  $\square$

**2.2. Analyzing the recursive equation.** Our goal now is to show that any random variable satisfying the recursive distributional inequality in Proposition 8 must be infinite with positive probability. With  $\epsilon$  as in Proposition 8, we introduce an operator  $\mathcal{A} = \mathcal{A}(\mu)$  that acts on nonnegative integer-valued random variables. Given such a random variable  $T$ , we let  $s \in S$  be sampled according to  $\mu$ , and  $X_{\epsilon} \sim \text{Bernoulli}(\epsilon)$ , both independent of one another. Also, we take  $T_i$  to be i.i.d. copies of  $T$ . We define a new random variable  $\mathcal{A}T$  to have distribution

$$\mathcal{A}T \stackrel{d}{=} \mathbf{1}\{s < s_2\}(X_{\epsilon}(T_1 + T_2) + (1 - X_{\epsilon})T_3) + \mathbf{1}\{s = s_2\}(T_4 + T_5) + \mathbf{1}\{s = s_1\}.$$

By Proposition 8, we have

$$(5) \quad \tau \succeq \mathcal{A}\tau.$$

We show that the probability that the first bullet with speed  $s_2$  survives,  $\mathbf{P}[\tau = \infty]$ , is positive in the following way. We first prove in Lemma 9 that  $\mathcal{A}$  is monotonic with respect to stochastic domination. Then, we show in Lemma 10 that  $\mathcal{A}^{\infty}\tau$  is equal to a unique fixed



point,  $\tau^*$ . Combine this with (5), and we have  $\tau \succeq \tau^*$ . Finally, we prove in Proposition 11 that any random variable fixed by  $\mathcal{A}$  is infinite with positive probability. From this it follows that  $\tau$  is infinite with positive probability. We go through this rigamarole with  $\tau^*$  because, just as in the warm-up argument with the return time of a biased random walk, we seek an exact solution to the generating function relationship.

**Lemma 9.** *If  $T \succeq T'$  then  $\mathcal{A}T \succeq \mathcal{A}T'$ .*

*Proof.* Follows from the canonical coupling which sets each  $T_i \geq T'_i$ .  $\square$

**Lemma 10.** *Let  $\mathcal{A}^n$  denote  $n$  iterations of  $\mathcal{A}$ . It holds that  $\mathcal{A}^\infty \tau \stackrel{d}{=} \tau^*$  with  $\tau^*$  unique and  $\tau^* \stackrel{d}{=} \mathcal{A}\tau^*$ .*

*Proof.* Let  $F_n(k) = \mathbf{P}[\mathcal{A}^n \tau \leq k]$  be the cumulative distribution function of  $\mathcal{A}^n \tau$ . By the previous lemma, we have  $\{F_n(k)\}_{n=0}^\infty$  is an increasing bounded sequence. Let  $F(k)$  denote its limit. The function  $F(k)$  is non-decreasing and belongs to  $[0, 1]$ . Thus,  $F(k)$  is the density function of some random variable  $\tau^*$ . This limiting random variable must be fixed by  $\mathcal{A}$  since an additional iteration  $\mathcal{A}(\mathcal{A}^\infty \tau)$  will not change the distribution.  $\square$

Note that  $\tau^* \succeq \mathbf{1}\{s = s_1\}$ , and so  $\tau^*$  is not identically 0. In fact, the following proposition shows that any fixed point of  $\mathcal{A}$  is infinite with positive probability.

**Proposition 11.** *With  $\epsilon$  as in Proposition 8, so long as  $\mu(S') > 0$  and  $(1 + \epsilon)\mu(S') + 2\mu(\{s_2\}) \geq 1$ , it holds that  $\mathbf{P}[\tau^* = \infty] > 0$ .*

*Proof.* Let  $f(x) = \mathbf{E}x^{\tau^*}$ . Denote  $f(1^-) = \lim_{x \rightarrow 1^-} f(x)$ . Since the coefficients of the power-series expansion of  $f$  are exactly the point probabilities of  $\tau^*$ , we have  $f(1^-) = 1 - \mathbf{P}[\tau^* = \infty]$ .

Let  $p_3 = \mu(S')$  and  $p_2 = \mu(\{s_2\})$ , and  $1 - (p_3 + p_2) = \mu(\{s_1\})$ . Using independence, we can write  $\mathbf{E}x^{\mathcal{A}\tau^*}$  in terms of  $f$  to obtain

$$f(x) = p_3(\epsilon f(x)^2 + (1 - \epsilon)f(x)) + p_2 f(x)^2 + (1 - (p_3 + p_2))x.$$

We can rewrite this as a quadratic equation in  $f(x)$

$$(6) \quad 0 = Af(x)^2 + Bf(x) + C$$

with

$$A = p_3\epsilon + p_2, \quad B = p_3(1 - \epsilon) - 1, \quad C = (1 - p_3 - p_2)x.$$

The discriminant,  $D(x) = B^2 - 4AC$ , is strictly decreasing, and thus minimized at  $x = 1$ . One can check that  $D(1) = ((1 + \epsilon)p_3 + 2p_2 - 1)^2$  is nonnegative if and only if

$$(1 + \epsilon)p_3 + 2p_2 \geq 1.$$

This is why we take this as a hypothesis for  $\mu$ . In this regime, (6) has exactly two solutions for  $0 \leq x \leq 1$ . We use the ‘+’ root, since  $f(0) = 0$ . The quadratic formula gives the closed form

$$f(x) = \frac{-B + \sqrt{D(x)}}{2A}.$$

Recall, we are interested in  $f(1^-)$ . It is straightforward to evaluate the above formula at  $x = 1$ . This yields

$$f(1^-) = \frac{p_2}{p_3\epsilon + p_2},$$

which is less than one so long as  $p_3 > 0$ . Hence the hypothesis  $\mu(S') > 0$ .  $\square$

The  $\epsilon$  in the above proposition is small and difficult to compute. Still, its existence expands on the set of  $(S, \mu)$  for which  $b_1$  survives. The hypotheses of Theorem 2 are weaker than what we prove. We state them as they are for the sake of simplicity. We are now ready to establish the result.

*Proof of Theorem 2.* By (5), Lemma 9, and Proposition 11,  $\tau$  is stochastically larger than a random variable that is infinite with positive probability. Hence  $\tau$  is infinite with positive probability.  $\square$

### 3. PERISHING OF SLOW BULLETS

In this section, we assume that  $(S, \mu)$  satisfies the hypotheses of Theorem 4. We start by describing the transformation from the bullet problem to the arrow process mentioned in the introduction. To facilitate this discussion, we use the notation  $a_i$  for the arrow at  $i \in \mathbb{Z}$ , and  $s(a_i)$  for its speed. Recall that each  $s(a_i)$  is i.i.d. in  $\mathbb{R}$ , and each arrow  $a_i$  begins moving simultaneously along  $\mathbb{R}$  at speed  $s(a_i)$ . As with bullets, collisions result in mutual annihilation. Moreover, we say an arrow process is *one-sided* if we have arrows only for  $i \in \mathbb{Z}^+ := \{1, 2, \dots\}$  or only for  $i \in \mathbb{Z}^- := \{-1, -2, \dots\}$ ; if we have arrows for all  $i \in \mathbb{Z}$ ; we say the process is *two-sided*.

**3.1. Equivalence to the arrow model.** Given a realization of the bullet speeds  $\vec{s} = (s(b_i))_{i=1}^\infty$  we can partition  $\mathbb{Z}^+$  into the indices of bullets that mutually annihilate one another, i.e., the indices of the bullets that perish:

$$B_{\vec{s}} = \left\{ \{i \in \mathbb{Z}^+ \text{ such that } b_i \mapsto b_k \text{ or } b_k \mapsto b_i\} : k \in \mathbb{Z}^+ \right\}.$$

Note this is the set of bullets that actually (as opposed to potentially) annihilate one another. We can define a similar collection for the indices of colliding arrows. Let  $\vec{r}$  be a realization of arrow speeds on  $\mathbb{Z}^+$ . Letting  $a_i \mapsto a_k$  denote annihilation of two arrows where  $i < k$ , define

$$A_{\vec{r}} = \left\{ \{i \in \mathbb{Z}^+ \text{ such that } a_i \mapsto a_k \text{ or } a_k \mapsto a_i\} : k \in \mathbb{Z}^+ \right\}.$$

We say that  $\vec{s}$  and  $\vec{r}$  are *annihilation-equivalent* if  $A_{\vec{r}} = B_{\vec{s}}$ . The following lemma gives a formula to translate between these two processes.

**Lemma 12.**  $\vec{s}$  and  $\vec{r} = (1/s(b_i))_{i=1}^\infty$  are annihilation-equivalent.

*Proof.* Consider a graphical representation of the bullet problem with the  $x$ -axis the time elapsed, and the  $y$ -axis the distance from the origin. Annihilations are then the same as the intersection of two lines. When we reflect across the line  $d = t$  we obtain an arrow process on  $\mathbb{Z}^+$  where the bullet  $b_i$  fired at time  $t = i$  now corresponds to the arrow  $a_i$  with speed  $1/s(b_i)$ . See Figure 4.  $\square$

We will also need to transform an arrow process on  $\mathbb{Z}^-$  into a bullet problem. Given  $\vec{r}^- = (s(a_i))_{i=-1}^{-\infty}$  we say that  $\vec{r}^-$  and  $\vec{s}$  are annihilation-equivalent if  $-A_{\vec{r}^-} = B_{\vec{s}}$ .

**Lemma 13.** Suppose that arrow speeds are in the interval  $[r_n, r_1]$  where  $r_n < r_1$  are fixed. Let  $\vec{r}^-$  be a realization of arrow speeds on  $\mathbb{Z}^-$  and let  $G(x) = (|r_n| + |r_1| - x)^{-1}$ . It holds that  $\vec{r}^-$  and a bullet process with realization  $(G(s(a_i)))_{i=-1}^{-\infty}$  are annihilation-equivalent.

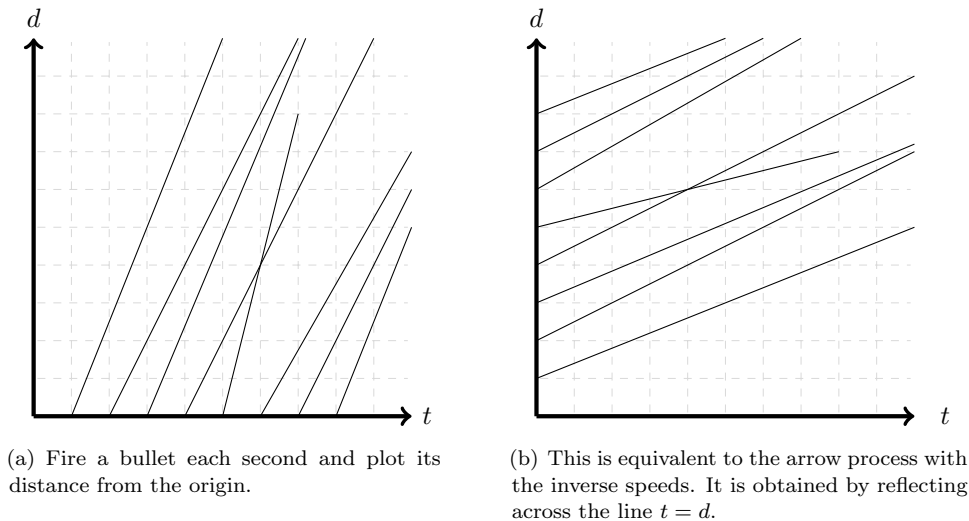


FIGURE 4. Transforming a bullet process into an equivalent arrow process.

*Proof.* Because arrow annihilations depend only on the relative speeds, we have the same annihilation behavior when we translate all speeds by the same constant. Shift each arrow by  $-(|r_n| + |r_1|)$ . When we reflect across  $d = t$  we obtain a bullet problem with positive speeds given by  $s(b_i) = G(s(a_i))$ . See Figure 5.  $\square$

**3.2. Proving Theorem 3 and Theorem 4.** The proofs that bullets perish are proofs by contradiction. When we assume a slow bullet survives, we show that this implies two arrow speeds survive with positive probability in a two-sided arrow process. This contradicts the following proposition that Yuval Peres, Alexandre Stauffer and Lorenzo Taggi shared with us.

**Proposition 14.** *In an arrow process on  $\mathbb{Z}$ , there can be at most one arrow speed that occurs in the process that survives with positive probability.*

*Proof.* Suppose there are two different arrow speeds, say  $r_i$  and  $r_j$  that both occur and have a positive probability of surviving. Translation invariance ensures the arrow process on  $\mathbb{Z}$  is ergodic. By the Birkhoff ergodic theorem, we have a positive fraction of arrows with these two different speeds that are never annihilated. This is a contradiction since there must be infinitely many surviving arrows with speed  $r_j$  to the left and to the right of any surviving arrow with speed  $r_i$ , and thus they must collide.  $\square$

We saw in Lemma 12 that we can transform a bullet process with bullet speeds distributed as some random variable  $X \in [s_n, s_1]$  to a one-sided arrow process on  $\mathbb{Z}^+$ . The arrow speeds are distributed as  $1/X$ . If we extend the arrow process to all of  $\mathbb{Z}$  and consider the arrows in  $\mathbb{Z}^-$ , then we obtain a transformed bullet process. By Lemma 12 and Lemma 13, the resulting bullet process has speeds distributed as  $T(X) = G(1/X)$ . We record a few facts about the transformation  $T$ . Each is elementary to verify by hand, so we omit the proof.

**Lemma 15.** *Let  $T: [s_n, s_1] \rightarrow [s_n, s_1]$  be given by  $T(x) = (s_n^{-1} + s_1^{-1} - x^{-1})^{-1}$ . The function  $T$  has the following properties:*

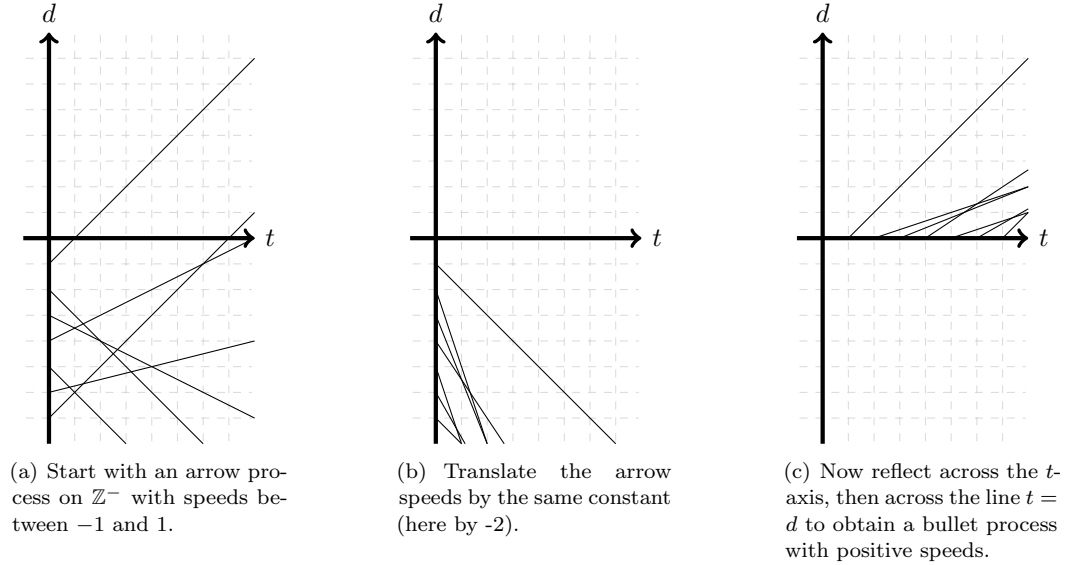


FIGURE 5. Transforming an arrow process on  $\mathbb{Z}^-$  into an equivalent bullet process.

- (i)  $T$  is decreasing.
- (ii)  $T(s^*) = s^*$  for  $s^* = \frac{2s_1s_n}{s_1+s_n}$ . This fixed point is unique.
- (iii)  $T^{-1}(x) = T(x)$ .
- (iv)  $T(s_1) = s_n$  and  $T(s_n) = s_1$ .

*Proof of Theorem 3.* Suppose that

$$\mathbf{P}[b_1 \text{ survives} \mid s(b_1) = y_2] = p > 0.$$

By monotonicity,

$$(7) \quad \mathbf{P}[b_1 \text{ survives} \mid s(b_1) = y_1] = q \geq p.$$

Transform the bullet process with  $s(b_1) = y_1$  to an arrow process on  $\mathbb{Z}^+$  as in Lemma 12. Now, extend the arrow process to all of  $\mathbb{Z}$ . Our hypothesis and equivalence in Lemma 12 ensures that the probability that arrow  $a_1$  is never annihilated by an arrow from the right is  $p$ .

The bullet process induced by the arrow process on  $\mathbb{Z}^- \cup \{0, 1\}$  has speed distribution  $T(s(b_1))$ . By Lemma 15, we know that  $T(s(b_1))$  is the second fastest speed in a bullet process satisfying the hypotheses of Theorem 2. It follows that  $a_1$  survives with some probability from the left.

We can repeat the same reasoning to deduce that arrows with speed  $1/y_2$  also survive with positive probability. Thus, we have two arrow speeds  $1/y_1$  and  $1/y_2$  that survive with positive probability in the arrow process on  $\mathbb{Z}$ . This contradicts Proposition 14.  $\square$

The proof of Theorem 3 used a generic invariance among bullet processes with two fastest speeds. The hypothesis on  $\mu$  in Theorem 4 introduces a special symmetry with respect to the induced two-sided arrow process. Essentially, the arrow process on  $\mathbb{Z}^+$  is the original

bullet problem, and the arrow process on  $\mathbb{Z}^- \cup \{0, 1\}$  is an inverted version of the original bullet process (i.e., the slow speeds are now fast). Our measure  $\mu$  is chosen so that the inverted model has the same measure on bullet speeds. When we assume an arrow with slow speed survives, this again gives two arrow speeds that survive with positive probability. The symmetry allows us to push this argument through for the slowest half of bullets.

*Proof of Theorem 4.* Let  $(S, \mu)$  be as in the hypotheses, and have  $X$  be distributed according to  $\mu$ . To show a contradiction suppose that

$$(8) \quad \mathbf{P}[b_1 \text{ survives} \mid s(b_1) < s^*] = p > 0.$$

Notice that by monotonicity of the bullet process, this implies that

$$(9) \quad \mathbf{P}[b_1 \text{ survives} \mid s(b_1) \geq s^*] = q \geq p.$$

Transform the bullet process conditioned on  $s(b_1) < s^*$  to an arrow process on  $\mathbb{Z}^+$  as in Lemma 12. Now, extend the arrow process to all of  $\mathbb{Z}$ . Our hypothesis and equivalence in Lemma 12 ensures that the probability  $a_1$  is never annihilated by an arrow from the right conditional on its speed being greater than  $1/s^*$  is  $p$ .

We claim that the requirement at (1) ensures that the probability that  $a_1$  is never destroyed by an arrow from the left is  $q$ . Indeed, Lemma 13 ensures that the arrow process restricted to  $a_1, a_0, a_{-1}, \dots$  is annihilation-equivalent to a bullet process with speeds distributed as  $T(s(b_1))$ . The facts that  $T = T^{-1}$  and that  $T$  is decreasing (see Lemma 15 (i) and (iii)) imply

$$\mathbf{P}[T(X) \leq x] = \mathbf{P}[X \geq T(x)] = \mu([T(x), s_1]).$$

By the hypothesis  $\mu([s_n, x]) = \mu([T(x), s_1])$ , we deduce that

$$\mathbf{P}[T(X) \leq x] = \mathbf{P}[X \leq x].$$

Thus,  $T(X)$  and  $X$  are identically distributed. This means that the induced arrow process from  $(S, \mu)$  on  $\mathbb{Z}^- \cup \{0, 1\}$  is also equivalent to an  $(S, \mu)$ -bullet process. However, the image of  $r_1$  in this new process has speed  $T(r_1)$ . By Lemma 15 (i) and (ii) we have  $T(s(b_1)) > s(b_1)$ . So, by (9), we have that  $a_1$  is not annihilated by any arrow from the left with probability  $q$ .

No bullets reaching  $a_1$  from the left, and no bullets reaching  $a_1$  from the right are independent events. This independence along with the previous paragraph ensures that  $a_1$  survives with probability  $pq > 0$ . By symmetry, arrows with speed less than  $s^*$  survive with positive probability (except the probability of survival from the right is now  $q$  and from the left is now  $p$ ). We then have different arrow speeds surviving with positive probability, but this contradicts Proposition 14.  $\square$

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